$X \quad p_X \qquad E[X] = \sum_{x \in \mathbb{R}} x \, p_X(x)$

$E[|X|^k] < +\infty \longrightarrow E[X^k] = \sum x^k p_X(x)$

X ha momento k-simo finito

$$E[\phi(X)] = \sum_{x \in \mathbb{R}} \phi(x) \, p_X(x)$$

$$E\left[|X - E[X]|^k \right]$$

Def : X v.a. discreta . se ha momento secondo finito , posso considerare

$$E\left[\left(X-E[X]\right)^2\right] = Var(X)$$

Varianza

X p_X

$$Var(X) = \sum_{x \in \mathbb{R}} \underbrace{\left(x - E[X]\right)^2}\, p_X(x)$$

$$Var(X) \geq 0 \longrightarrow \sqrt{Var(X)}$$

deviazione standard.

PROPRIETÀ DELLA VARIANZA

- $Var(X) = E[(X - E[X])^2] = E[X^2 + (E[X])^2 - 2XE[X]] =$

$= E[X^2] + E[(\underline{E[X]})^2] + E[\underline{-}\,2X\underline{E[X]}] =$

$= E[X^2] + (E[X])^2 - 2(E[X])^2 \cancel{E[X]} =$

$= E[X^2] - (E[X])^2$

- $Var(cX) = E[(cX - E[cX])^2] =$ ← def

$= E[(cX)^2] - (E[cX])^2 =$ ← formula alternativa

$= E[c^2X^2] - (cE[X])^2 =$

$= c^2E[X^2] - c^2(E[X])^2 =$

$= c^2(E[X^2] - (E[X])^2) = c^2 Var(X)$

$Var(-X) = Var(X)$

- $\mathrm{Var}(X+c) = E[(X+c-E[X+c])^2] =$

$$= E[(X+\cancel{c}-E[X]-\cancel{c})^2] =$$

$$= E[(X-E[X])^2] = \mathrm{Var}(X)$$

- $\mathrm{Var}(X) = 0 \Leftrightarrow X = cst$

$\Leftarrow$ X cst $X = c$ $E[X] = c$ $E[(X-E[X])^2] =$

$$= E[(c-c)^2] = 0$$

$$\Rightarrow \mathrm{Var}(X) = 0 \Rightarrow X = cst$$

$$\mathrm{Var}(X) = \sum_{x \in \mathbb{R}} (x - E[X])^2 p(x) = 0$$

$$\Updownarrow$$

$$(x - E[X])^2 \underbrace{p(x)}_{} = 0 \qquad \forall x \in \mathbb{R} \quad x = E[X]$$

$$\Downarrow$$

$$X = cst = E[X]$$

Teo Tchebyceff: X v.a. con varianza finita, allora $\forall \varepsilon > 0$ vale

$$P(|X - E[X]| > \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$$

Teorema di Markov: Y v.a. con speranza finita, allora $\forall \varepsilon > 0$

$$P(|Y| > \varepsilon) \leq \frac{E[|Y|]}{\varepsilon} \Longleftrightarrow \boxed{E[|Y|] \geq \varepsilon P(|Y| > \varepsilon)}$$

$$P(|X - E[X]| > \varepsilon) = P((X - E[X])^2 > \varepsilon^2) \leq$$

$$\leq \frac{E[(X - E[X])^2]}{\varepsilon^2} = \frac{Var(X)}{\varepsilon^2}$$

$P(|Y| > 3)$

$\boxed{Y^2 > 9} \Longleftrightarrow \boxed{|Y| > 3}$

$y^2 > 9 \Longleftrightarrow |y| > 3$

$\mathcal{B}(1,p)$ Bernoulliana

$X \begin{cases} 1 \\ 0 \end{cases}$ $p = P(X=1)$ $1-p = P(X=0)$

$E[X] = p$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = p$$

$$Var(X) = p - p^2 = p(1-p)$$

$X^2 \begin{cases} 1 \\ 0 \end{cases}$

$P(X^2=1) = P(X=1) = p$

$P(X^2=0) = P(X=0) = 1-p$

$X \sim \mathcal{P}(\lambda) \qquad E[X] = \lambda \qquad Var(X) = E[X^2] - (E[X])^2$

$$E[X^2] = \sum_{k=0}^{\infty} k^{\cancel{2}} e^{-\lambda} \frac{\lambda^k}{\cancel{k!}_{\,(k-1)!}} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} = \qquad k-1 = h$$

$$= e^{-\lambda} \lambda \sum_{h=0}^{\infty} (h+1) \frac{\lambda^h}{h!} = \lambda \boxed{\sum_{h=0}^{\infty} (h+1) e^{-\lambda} \frac{\lambda^h}{h!}} =$$

$$= \lambda E[X+1] = \lambda (E[X] + 1) =$$

$$= \lambda^2 + \lambda$$

$$Var(X) = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

Geometrica

$$Var(X) = E[X^2] - (E[X])^2 = \sum_{k=1}^{\infty} k^2 p(1-p)^{k-1} - \frac{1}{p^2} = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

$$X \sim B(n,p) \qquad X = \sum_{i=1}^{n} X_i \qquad X_i \sim B(1,p)$$

indipendenti

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right)$$

COVARIANZA.

$E[XY] = E[X]E[Y] \leftarrow$ vero se X, Y sono indipendenti

Def X, Y v.a.

$$Cov(X,Y) = E\left[(X - E[X]) \cdot (Y - E[Y])\right]$$

$$Cov(X,Y) \le \sqrt{Var[X]} \cdot \sqrt{Var[Y]}$$

p_{XY}

$$Cov(X,Y) = \sum_{x,y} (x - E[X])(y - E[Y]) \cdot p_{XY}(x,y)$$

$$\mathrm{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])] =$$

$$= E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$

$$= E[XY] - E[Y]E[X] - \cancel{E[Y]E[X]} + \cancel{E[X]E[Y]}$$

Def: diremo che due variabili aleatorie X, Y sono non correlate se la $\mathrm{Cov}(X,Y) = 0$

Se X, Y sono ind. $\Rightarrow E[XY] = E[X]E[Y] \Rightarrow \mathrm{Cov}(X,Y) = 0$

$\Rightarrow X, Y$ sono non correlate.

$$X \begin{cases} -1 & \frac{1}{3} \\ 0 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{cases} \qquad Y = X^2 \qquad E[X] = -1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$$

$$X^2 \begin{cases} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{cases} \qquad P(X^2=1) = P(X=1 \cup X=-1) = \frac{2}{3}$$

$$\mathrm{Cov}(X,Y) = E[X\cdot X^2] - \underbrace{E[X]\cdot E[X^2]}_{0} = E[X^3] = (-1)^3\cdot\frac{1}{3} + 0^3\cdot\frac{1}{3} + 1^3\cdot\frac{1}{3} = 0$$

$$\to 0$$

$$P(X=1, Y=0) = P(X=1, X^2=0) = 0$$

$$P(X=1)\cdot P(X^2=0) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$$

$\cancel{X} \quad Y$

$$\mathrm{Var}(X+Y) = E\left[\left(X+Y-E[X+Y]\right)^2\right] =$$

$$= E\left[\left(\left(X-E[X]\right) + \left(Y-E[Y]\right)\right)^2\right] =$$

$$= E\left[\left(X-E[X]\right)^2 + \left(Y-E[Y]\right)^2 + 2\left(X-E[X]\right)\left(Y-E[Y]\right)\right]$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + \underline{\underline{2\,\mathrm{Cov}(X,Y)}}$$

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \Leftrightarrow X, Y$ sono non correlate

$$X = \sum_{i=1}^{n} X_i \qquad X_i \text{ indipendenti}$$

$$\operatorname{Var} X = \operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$$

2 telefoni

Mediamente al primo telefono arrivano 5 telefonate
al secondo 3 telefonate.

Qual è la probabilità che in tutto arrivino 10 telefonate

$X \sim \mathcal{P}(5) \qquad Y \sim \mathcal{P}(3) \qquad X+Y$

$$P(X+Y=10) = P\left(\bigcup_{i=0}^{10} (X=i,\ Y=10-i)\right) = \sum_{i=0}^{10} P(X=i,\ Y=10-i)$$

$$= \sum_{i=0}^{10} P(X=i)\,P(Y=10-i) = \sum_{i=0}^{10} e^{-5}\frac{5^i}{i!} \cdot e^{-3}\frac{3^{10-i}}{(10-i)!} =$$

$$= \frac{e^{-(5+3)}}{10!} \boxed{\sum_{i=0}^{10} \frac{10!}{i!\,(10-i)!}\, 5^i\, 3^{10-i}} = \frac{e^{-(5+3)}}{10!} \sum_{i=0}^{10} \binom{10}{i} 5^i 3^{10-i}$$

$$= \frac{e^{-8}}{10!}(5+3)^{10} = \frac{e^{-8}}{10} 8^{10}$$

$X+Y \sim \mathcal{P}(3+5)$